

RG flow on random surfaces with handles and closed string field theory

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Abstract

The renormalization group flow in two-dimensional field theories that are coupled to gravity has unusual features: First, the flow equations are second order in derivatives. Second, in the presence of handles the flow has quantum mechanical properties. Third, the beta functions contain the elementary higher-genus vertices of closed string field theory. This is demonstrated at simple examples and is applied to derive various results about gravitationally dressed beta functions. The possibility of interpreting closed string field theory as the theory of the renormalization group on random surfaces with random topology is considered.

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1. Introduction

Recently there have been several studies of the flow of coupling constants in two-dimensional field theories that are coupled to gravity [1, 2, 3]. Among the results are the phase diagram of the sine-Gordon model coupled to gravity [4] and the gravitational dressing of beta function coefficients for bosonic [4, 5, 6, 7, 8] and supersymmetric theories [9]. These results can be obtained either in light-cone gauge or in conformal gauge. While the light-cone gauge approach is more rigorous and uses standard techniques of field theory, the conformal gauge approach seems to always yield the same results with less effort. In this paper it is shown that it can also be easily extended to higher-genus surfaces.

We consider two-dimensional conformal field theories perturbed by scaling operators Φ_i with coupling constants λ^i . The action is

$$S = S_{cft} + \lambda^i \int d^2\xi \sqrt{g} \Phi_i. \quad (1.1)$$

Without gravity, the flow of coupling constants is then described by the flow equations

$$\dot{\lambda}^i(\tau) = \beta^i(\lambda^j) \quad (1.2)$$

with beta functions β^i and “renormalization group time” τ . When gravity is “turned on” and the world-sheet topology is allowed to fluctuate, this flow is modified. Three types of modifications will be discussed here:

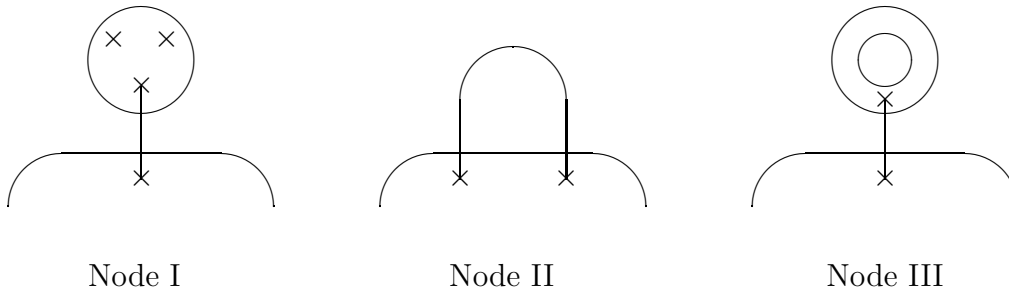
1. The time-derivative in (1.2) is replaced by a simple second-order derivative operator.¹
2. Classical flow trajectories $\lambda^i(\tau)$ are replaced by “wave packets” that can spread in the space of coupling constants. Equivalently, the λ^i are replaced by quantum operators.
3. The beta functions on the right-hand side of (1.2) are modified by the elementary higher-genus vertices of closed string field theory.

These modifications will be derived and discussed at the examples of the minimal models and the $c = 1$ model on a circle up to cubic order in λ, κ , where κ^2 is the topological coupling constant. Our motivation for studying the renormalization group flow on random surfaces is two-fold. First, we hope that the results about phase diagrams and gravitationally dressed coefficients mentioned above will be followed by similar results about the critical behavior of other, more physical systems of random surfaces, like perhaps the 3d Ising model or QCD.

Second, points 2 and 3 extend to higher genus the observation [10, 11, 12, 13, 14] that renormalization group trajectories in $2d$ field theories coupled to gravity can be regarded as classical solutions of string theory, where the λ^i represent the target space fields, τ is related to time and the flow equations are related to the string equations of motion (hence point 1). Reversing the viewpoint, one is tempted to *define* perturbative string theory in terms of the flow on (super) random surfaces. Our hope is that this definition, which is made more explicit towards the end of this paper, can be naturally extended to a nonperturbative one.

¹This has been pointed out previously e.g. in [10].

The line of argument in this paper is as follows. Coupling constants flow because of logarithmic divergences that make the renormalized coupling constants sensitive to the cutoff scale. As is well-known, such divergences come from the boundary of moduli space, i.e., from pinched surfaces whose pinching radius r is restricted by the short-distance cutoff a on the world-sheet: $r \geq a$. Three types of pinched surfaces (nodes) can be distinguished:



Each “ \times ” represents an operator insertion and each line a pinch (the ‘ring’ in the last diagram represents a torus). Node I is already present on genus zero surfaces. In the absence of gravity, it leads to the standard quadratic beta function coefficients. In section 2 it is first reviewed how gravity modifies these coefficients in the conformal gauge approach. The relation with the string equations of motion is then explained following [15] and is used to derive the modification of cubic beta function coefficients by gravity.

Node II represents a pinched handle. As is demonstrated in section 3, all scale dependence coming from massive modes (as defined later) propagating through node II can be absorbed in a running topological coupling constant. But massless modes lead to a bilocal scale dependence that can only be absorbed by “quantizing” the flow (the precise meaning of this will become clear). Some observations about various issues including background (in) dependence are also made in section 3.

Node III is responsible for the well-known Fischler-Susskind mechanism [16]. In section 4 its effect on the flow of the radius in the $c = 1$ model on a circle is discussed. It is pointed out that the corresponding poles precisely agree with those seen in matrix model amplitudes. Now, the scale dependence of part of node III is already absorbed by quantizing the flow to account for node II, as seen in section 5. Only the remainder, an elementary genus-one closed string field vertex, then contributes to the beta functions.

Section 6 contains some suggestions about the flow in supersymmetric theories with central charge $\hat{c} \geq 9$ coupled to supergravity, about the possibility of tunneling of the flow and about viewing closed string field theory as the theory of the flow in unitary theories on random surfaces. Section 6 also contains a summary of our results.

For simplicity, in this paper attention is restricted to states with equal left- and right-moving momenta (thus excluding winding modes) that also do not contain ghost operators. Also, we will not worry about possible contributions of gravitational descendents from boundaries of moduli space, since it is expected (from BRST invariance in string theory) that the sum of these contributions cancels. It is understood that a rigorous discussion should include an analysis of the full BRST cohomology and its interactions in the spirit of [17, 18]. However, we do not expect that this will modify our basic conclusions.

2. Node I and second-order flow equations

In this section it is reviewed how the standard renormalization group flow is modified by gravity on a genus zero surface. The “gravitational dressing” of cubic beta function coefficients is also derived here, in qualitative agreement with the independent result [8].

2.1. Renormalization group flow in conformal gauge

Since conformal gauge will be used to study the flow, let us first recall at the example of node I how the dressing of beta functions is derived in this gauge. It is done in two steps. First, the effective action for the conformal factor is constructed. Then constant shifts of the conformal factor are absorbed in running coupling constants. It will be assumed that the cosmological constant, which is small in the ultraviolet, does not affect the short-distance effects that are responsible for the flow of coupling constants. This leads to agreement with matrix model results (see below).

Theory (1.1) coupled to gravity can be described in conformal gauge by a conformal field theory with an additional field, the conformal factor ϕ of the world-sheet metric. The first step is to write down the most general local renormalizable action for the combined theory, order by order in the coupling constants λ^i [3] (setting $\alpha' = 2$):

$$S = S_{\text{cft}} + \frac{1}{8\pi} \int d^2\xi \sqrt{\hat{g}} \{ (\partial\phi)^2 - QR^{(2)}\phi + \text{ghosts} \} \quad (2.1)$$

$$+ \lambda^i \int \Phi_i e^{\alpha_i \phi} \quad (2.2)$$

$$+ \lambda^j \lambda^k \int X_{jk} \quad (2.3)$$

$$+ \lambda^j \lambda^k \lambda^l \int X_{jkl} + \dots \quad (2.4)$$

Here, $X_{jk}(\phi), X_{jkl}(\phi)$ are operators to be determined below. \hat{g} is a fictitious, arbitrarily chosen background metric that nothing physical can depend on. In particular, the combined theory, including all of its correlation functions must be scale invariant. This is the guiding principle that determines the coefficients Q , α_i and the operators X_{jk}, X_{jkl} order by order in λ : to zeroth order, scale invariance determines

$$Q = \frac{1}{3} \sqrt{25 - c}$$

to make the total central charge zero. To linear order in λ , scale invariance requires

$$\alpha_i^\pm = -\frac{Q}{2} \pm \omega_i \quad \text{with} \quad \omega_i = \sqrt{h_i - 2 + \frac{Q^2}{4}}, \quad (2.5)$$

where h_i is the scaling dimension of Φ_i , such that the operators

$$V_i \equiv \Phi_i e^{\alpha_i^+ \phi}, \quad \bar{V}_i \equiv \Phi_i e^{\alpha_i^- \phi} = \Phi_i e^{(-Q - \alpha_i^+) \phi}$$

have dimension two. ω_i can be thought of as (imaginary) frequency. The operator \bar{V}_i is usually assumed to “not exist” [19] and will be ignored here.² If Φ_i is almost marginal, $\omega_i \sim \frac{Q}{2}$ and $\alpha_i^+ \sim 0$. To quadratic order in λ , scale invariance then determines X_{jk} , which in this case is a universal operator. One finds [4]:

$$X_{jk} = \frac{\pi}{2\omega_i} c_{jk}^i \phi V_i. \quad (2.6)$$

Indeed, this term is needed to ensure scale invariance of the two-point function, i.e., of the second derivative of the partition function with respect to the coupling constants:

$$\frac{\delta^2}{\delta\lambda^j\delta\lambda^k} Z \sim \langle e^{-S} \int V_j \int V_k \rangle - \frac{\pi}{2\omega_i} c_{jk}^i \langle e^{-S} \int \phi V_i \rangle. \quad (2.7)$$

Namely, the short-distance singularity in the operator product expansion (OPE) of V_j with V_k must be regularized, e.g. by setting a minimal distance \hat{a} between the operators. Then the first term in (2.7) contains a divergent part

$$- \pi c_{jk}^i (\log \hat{a}) \langle e^{-I} \int V_i \rangle \quad (2.8)$$

that depends on the fictitious scale through the fictitious cutoff \hat{a} . Since

$$(L_0 + \bar{L}_0 - 2) \phi V_i = -2\omega_i V_i, \quad (2.9)$$

the second term in (2.7) exactly cancels this scale dependence ($L_0 + \bar{L}_0 - 2$ is the generator of scale transformations). This situation, where V_j approaches V_k , is conformally equivalent to node I, where a long cylinder of length $L \sim \log \hat{a}$ connects two separate surfaces. For $\alpha_i = 0$, the interaction terms (2.2+2.3) become

$$\lambda^i \Phi_i + \frac{\pi}{Q} c_{jk}^i \lambda^j \lambda^k \Phi_i \phi. \quad (2.10)$$

The second step is to read off the renormalization group flow “in the presence of gravity” from the action. By construction, there is no flow with respect to the fictitious background scale defined by \sqrt{g} . But since $\sqrt{g} = \sqrt{\hat{g}} e^{\alpha\phi}$ defines the physical scale, a constant shift

$$\phi \rightarrow \phi + \frac{2}{\alpha} \tau, \quad \alpha = -\frac{Q}{2} + \sqrt{\frac{Q^2}{4} - 2} \quad (2.11)$$

corresponds to a scale transformation or – more precisely – to a rescaling of the *physical* cutoff $a \rightarrow ae^\tau$. The crucial point here is that ϕ lives on a half-line, bounded by the physical cutoff: $\phi \leq (\log a)/\alpha$ [11]. Otherwise a constant shift of ϕ would be trivial since ϕ is integrated over. Such a shift of ϕ can now be absorbed in running coupling constants $\lambda^i(\tau)$, where $\tau \rightarrow \infty$ corresponds to the ultraviolet. E.g., in (2.10) the shift (2.11) induces

$$\tau \frac{2\pi}{\alpha Q} c_{jk}^i \lambda^j \lambda^k \Phi_i.$$

² \bar{V}_i will play a role in the presence of handles, as will be seen in later sections.

This term can be absorbed in (2.10) to order λ^2 by defining [4]

$$\lambda^i(\tau) \sim -\tau \frac{2\pi}{Q\alpha} c_{jk}^i \lambda^j \lambda^k \rightarrow \dot{\lambda}^i \sim -\frac{2}{Q\alpha} \pi c_{jk}^i \lambda^j \lambda^k. \quad (2.12)$$

By comparison, the flow without gravity is given by

$$\dot{\lambda}^i \sim \pi c_{jk}^i \lambda^j \lambda^k + \pi d_{jkl}^i \lambda^j \lambda^k \lambda^l + \dots \quad (2.13)$$

The “gravitational dressing factor” $-2/(Q\alpha)$ in (2.12) of the universal quadratic term indeed agrees with the light cone gauge result [5]. The method presented here can also be used to derive the phase diagram of the sine-Gordon model coupled to gravity [4], in agreement with matrix model results [20, 21].

2.2. Dressing of the λ^3 coefficients

To determine the gravitational dressing of the cubic beta function coefficients d_{jkl}^i , it is useful to note that the scale invariance conditions for the effective action (2.1–2.4) are just the string equations of motion [22, 23] with ϕ playing the role of (euclidean) time.

Here it is assumed that the two-dimensional matter theory can be formulated as a sigma model; the dressed “matter” operators Φ_i then correspond to target space gravitons or almost marginal tachyon perturbations. E.g., the q -th minimal model with large q (large q ensures the existence of almost marginal perturbations) can be described by a scalar field x with a Landau-Ginzburg potential $T_q(x)$ [24]. Perturbations around the fixed point can be expanded in a complete set of scaling operators $T_{q,k}(x)$. After coupling to gravity one obtains a conformal field theory with x and ϕ as target space coordinates, a dilaton background Φ and a tachyon background T that can be expanded as

$$T(x, \phi) = T_q(x) + \lambda^k(\phi) T_{q,k}(x).$$

The trajectory $\lambda^k(\phi)$ then describes a classical string solution that asymptotically approaches the static solution $\lambda^k(\phi) = 0$. Graviton perturbations $G_{\mu\nu}(x, \phi)$ can be treated similarly if one picks the gauge $G_{\phi\phi} = 1, G_{x\phi} = 0$ as in (2.1–2.4). Expanding the string equations of motion in λ^i , one finds the universal form (setting $\alpha' = 2$) [15]:

$$O(\dot{\lambda}^2) + \ddot{\lambda}^i + Q\dot{\lambda}^i = \beta^i, \quad Q^2 = \frac{25-c}{3} + O(\lambda^2). \quad (2.14)$$

Here, the dot means derivative with respect to ϕ . The first equation is the graviton or tachyon equation while the second equation is the equation for the (shifted) dilaton zero mode, i.e., the x -independent part $\varphi_0(\phi)$ of the (shifted) dilaton $\varphi(x, \phi) \equiv 2\Phi - \sqrt{G}$ (Q is defined as $-\dot{\varphi}_0(\phi)$; to order λ^2 , $\varphi(x, \phi) \sim \varphi_0(\phi) \sim -Q\phi$; at higher orders, Q depends on ϕ). β^i are the *exact* beta functions in (1.2) of the theory without gravity, that is, the graviton and tachyon beta functions of the sigma model without the additional target space coordinate ϕ . (2.14) describes the damped/antidamped motion of a particle in theory space with the beta functions as a driving force. We refer to [15] for details.³

³(2.14) has only been derived for graviton- and B -field perturbations there, but the extension to tachyon perturbations is straightforward.

Now, ϕ is related to renormalization group time τ by the x -independent part of the tachyon $T(x, \phi) = T(\phi)$, which is the dressed area operator (see previous subsection and, e.g., [10, 11, 12, 41]). For the $c = 1$ model, the relation between ϕ and τ might be nonlinear (see [11]), but for the minimal models ($c < 1$), $T(\phi)$ has the simple form

$$T(\phi) \sim \mu \{e^{\alpha\phi} + c_{\mu k}^i \times O(\lambda) + O(\lambda^2)\},$$

where the second term corresponds to possible corrections of order $\mu\lambda$ that might arise if there are nontrivial OPE coefficients $c_{\mu k}^i$ in the OPE of the cosmological constant with one of the almost marginal operators V_k . However, those coefficients are known to be zero for our models. Therefore

$$\tau \sim -\frac{\alpha}{2}\phi + O(\lambda^2). \quad (2.15)$$

With this identification, the string equations of motion become the renormalization group flow equations. If $\dot{\lambda}$ is of order λ^2 (as in (2.13) and as will be assumed below), the $\dot{\lambda}^2$ terms in (2.14) are of order λ^4 and can be ignored at cubic order. Likewise, the $O(\lambda^2)$ terms in Q and the $O(\lambda^2)$ terms in (2.15) can also be ignored in the first equation of (2.14). Thus, at least for $\kappa^2 = 0$ and $c < 1$, gravity simply modifies the flow equation (2.13) to the second order differential equation

$$\frac{\alpha^2}{4}\ddot{\lambda}^i - \frac{\alpha}{2}Q\dot{\lambda}^i = \pi c_{jk}^i \lambda^j \lambda^k + \pi d_{jkl}^i \lambda^j \lambda^k \lambda^l + \dots, \quad (2.16)$$

up to nonuniversal higher-order terms.⁴ This is the precise form of the statement in point 1 of the introduction. Now, one is interested in solutions of (2.16) that also obey a first-order flow equation

$$\dot{\lambda}^i = \pi \tilde{c}_{jk}^i \lambda^j \lambda^k + \pi \tilde{d}_{jkl}^i \lambda^j \lambda^k \lambda^l + \dots, \quad (2.17)$$

where one can call \tilde{c}, \tilde{d} “modified beta function coefficients”. This ansatz guarantees that the flow without gravity is recovered in the “classical limit” of infinite negative central charge. Differentiating (2.17), plugging it into (2.16) and comparing the quadratic and cubic coefficients yields the “gravitationally dressed” coefficients:

$$\tilde{c}_{jk}^i = -\frac{2}{\alpha Q} c_{jk}^i, \quad \tilde{d}_{jkl}^i = -\frac{2}{\alpha Q} (d_{jkl}^i - \frac{2\pi}{Q^2} c_{jm}^i c_{kl}^m). \quad (2.18)$$

The first result is the same as before. For the case of only one coupling constant, where the coefficient d is universal, the second result agrees qualitatively with a recent calculation of Dorn [8].⁵ Since higher order beta function coefficients are not universal, all information about the gravitational dressing of beta function coefficients on a genus zero surface, at least in the vicinity of fixed points with $c < 1$, is thus encoded in the replacement

$$\dot{\lambda}^i \rightarrow \frac{\alpha^2}{4}\ddot{\lambda}^i - \frac{\alpha}{2}Q\dot{\lambda}^i$$

⁴A possible small dimension ϵ_i of λ^i can be absorbed in the λ corresponding to the kinetic term for x .

⁵However, after fixing normalizations the results seem to differ by a factor 2. This might signal a scheme dependence of \tilde{d} . I thank H. Dorn for pointing this out to me (see ref. [48]) after this paper appeared.

in the flow equation (1.2).

Next, one must ask whether this second order differential operator is modified at genus-one level. In general, there are indeed terms of the form

$$\kappa^2 \dot{\lambda}, \quad \kappa^2 \ddot{\lambda}, \quad \kappa^2 \dot{\lambda}^2 \quad (2.19)$$

in the string equations of motion, corresponding to terms like $\kappa^2 R$ (R is the target space curvature) in the string effective action. However, in the case discussed here, $\dot{\lambda}$ and $\ddot{\lambda}$ are at most of order λ^2 or of order κ^2 . Therefore the terms (2.19) are not relevant in the present discussion, where at most cubic orders in the simultaneous expansion in λ and κ are considered. So in the following only the right-hand-side of (2.16) will be modified by topology fluctuations.

3. Node II and quantum mechanical flow

In this section the scale dependence induced by node II is discussed to order κ^2 . Again, it must be remembered that there are two scales: the fictitious background scale defined by $\sqrt{\hat{g}}$ and the physical scale defined by $\sqrt{\hat{g}}e^{\alpha\phi}$. Node II must be and will be seen to be invariant under rescaling of \hat{g} , but not under physical scale transformations, corresponding to shifts of the Liouville mode ϕ .

Both the positively and the negatively dressed operators V_i and \bar{V}_i must be used in this section, since they correspond to creation and annihilation operators in string theory (see below). The Liouville dressing of \bar{V}_i grows faster than $e^{-\frac{Q}{2}\phi}$ in the infrared $\phi \rightarrow -\infty$. We believe that this causes no problem at least as long as this operator acts only on states with sufficiently positive Liouville momentum, so that it never creates states that “do not exist” in Liouville theory [19]. Note also that negatively dressed operators can apparently be seen and studied in the matrix models [25].

3.1. Massive modes

The question is whether the integration over thin handles leads to new divergences that induce new dependence on physical scale transformations $\phi \rightarrow \phi + \frac{2}{\alpha}\tau$. To study this, handles can be “integrated out”. This leads to a bilocal operator insertion on the surface, whose behavior under constant shifts of ϕ can then be studied.

By “integrating out handles”, the following is meant: surfaces are decomposed into elementary vertices and propagators (cylinders) as in string field theory [26], and cylinders connecting a surface with itself are replaced by bilocal operator insertions. It must be emphasized that “integrating out handles” does not mean reducing higher-genus correlation functions to genus-zero correlation functions, since, e.g., not all genus-1 surfaces can be regarded as a genus-0 surface plus a propagator. One still has to integrate over genus-1 surfaces corresponding to the elementary genus-1 string field vertices. But this will not lead to any new logarithmic dependence on the world-sheet cutoff, so it will not modify the flow of coupling constants.

To replace a cylinder by a bilocal operator insertion, one picks a complete set of off-shell states propagating through the cylinder, consisting e.g. of all scaling operators $\Phi_i(x)$ of the matter theory, properly normalized and dressed with *arbitrary* Liouville momenta. Then the cylinder (\sim node II) can be replaced by the bilocal insertion of *off-shell* operators

$$\kappa^2 \sum_i \int \frac{d\epsilon}{2\pi} \frac{1}{(L_0 + \bar{L}_0 - 2)_{i,\epsilon}} \int d^2z \Phi_i(z) e^{\epsilon\phi(z)} \int d^2w \Phi_i(w) e^{(-Q-\epsilon)\phi(w)}$$

(such that a handle adds the amount $-Q$ to the Liouville momentum) with inverse propagator

$$(L_0 + \bar{L}_0 - 2)_{i,\epsilon} = -(\epsilon + \frac{Q}{2})^2 + \omega_i^2.$$

The “frequency” ω_i has been defined in (2.5). In a standard fashion, this sum over off-shell states propagating through the cylinder can be replaced by a sum over *on-shell* states by either using Feynman’s tree theorem (see, e.g., [27]) or doing the contour integral over macroscopic [28] Liouville momenta ϵ . This yields in the case of massive modes (“massive” means $\omega_i > 0$) an insertion

$$\sum_i \frac{\kappa^2}{2\omega_i} \int d^2z V_i(z) \int d^2w \bar{V}_i(w). \quad (3.1)$$

Since V_i, \bar{V}_i are marginal, (3.1) is invariant under background scale transformations as it must be, except for the situation where V_i and \bar{V}_i coincide; this will be taken care of in the next section. Under constant shifts $\phi \rightarrow \phi + \frac{2}{\alpha}\tau$, (3.1) is multiplied by $\exp\{(\alpha_i^+ + \alpha_i^-)\frac{2}{\alpha}\tau\} = \exp\{-\frac{2}{\alpha}Q\tau\}$. Thus, all scale dependence that comes from massive states ($\omega_k > 0$) propagating through node II can be absorbed in a running topological coupling constant

$$\kappa^2 = \kappa^2(\tau) = \kappa_0^2 \exp\{\frac{2Q}{\alpha} \tau\}, \quad \kappa_0^2 \equiv \text{value at } \{\tau = 0\}. \quad (3.2)$$

Note that the “dressed” dimension $|\frac{2Q}{\alpha}|$ of the string coupling constant diverges in the weak gravity limit $c \rightarrow -\infty$, and that κ^2 is dimensionless for $c = 25$.

3.2. Massless modes

Let us now consider isolated “massless” modes ($\omega_k = 0$) that propagate through node II. Massless modes correspond to dressed matter primary fields Φ_k with dimensions

$$h_k = 2 - \frac{Q^2}{4}. \quad (3.3)$$

Actually, in the $c \leq 1$ models there is only one example – the cosmological constant in the $c = 1$ model on a circle. $\Phi_k \propto 1$ in this case. But the following discussion will be kept general, since it should also apply to more general models like supersymmetric theories with $\hat{c} = 9$ that may contain other isolated states with $\omega_k = 0$.

Taking the limit $\omega_k \rightarrow 0$ in (3.1) yields a divergent factor ω_k^{-1} . However, it must be remembered that ω_k is the Liouville momentum of the state $|k\rangle$ and the Liouville mode

lives in a box $\phi_{ir} \leq \phi \leq \phi_{uv}$. Here, ϕ_{ir} is the world-sheet infrared cutoff $-(\log \mu)/\alpha$ (whose value does not matter) and $\phi_{uv} = (2 \log a)/\alpha$ is the world-sheet ultraviolet cutoff. Therefore ω_k cannot quite become zero; instead, it should be replaced by its smallest possible value as in [29], i.e.

$$\frac{1}{\omega_k} \rightarrow \sim |\phi_{ir} - \phi_{uv}| = \frac{1}{\alpha}(2 \log a + \log \mu). \quad (3.4)$$

The overall coefficient, which is not reliably fixed by this qualitative argument, will be determined later. So from taking the limit $\omega_k \rightarrow 0$ in (3.1) one learns that, under a rescaling $a \rightarrow ae^\tau$ of the physical cutoff, isolated states with $\omega_k = 0$ propagating through node II lead to an operator insertion

$$\sim \tau \frac{\kappa^2}{\alpha} \int V_k \int V_k \quad \text{with} \quad V_k = \Phi_k e^{-\frac{Q}{2}\phi}. \quad (3.5)$$

This bilocal scale dependence is independent of the value of μ . It has its origin in the fact that ϕ is bounded by the physical cutoff a . κ^2 depends on τ as in (3.2) to absorb the constant shift of ϕ in V_k that comes with the rescaling of the cutoff. Note that there is no dependence on the fictitious cutoff \hat{a} , which can be taken all the way to zero: the integral over the node length l decays exponentially though slowly (at rate $\sim |\log a|^{-1}$) and thus need not be cut off by $\log \hat{a}$.⁶

The result (3.5) is also plausible from a different (though related) viewpoint. In the case $\omega_k = 0$, the two conjugate dressings of the operator Φ_k are $e^{-\frac{Q}{2}\phi}$ and $\phi e^{-\frac{Q}{2}\phi}$. This suggests that integrating out node II produces a term proportional to

$$\kappa^2 \int \Phi_k e^{\frac{Q}{2}\phi} \int \Phi_k \phi e^{\frac{Q}{2}\phi},$$

plus possibly a divergent term proportional to $\kappa^2 \int \Phi_k e^{\frac{Q}{2}\phi} \int \Phi_k e^{\frac{Q}{2}\phi}$ that transforms trivially under constant shifts of ϕ . Shifting ϕ then yields a scale dependence of the form (3.5).

Clearly, it is not possible to absorb the bilocal insertion (3.5) in a running coupling constant $\lambda^k(\tau)$. But one can consider a Gaussian distribution with width σ in the space of theories parametrized by λ^k (compare e.g. with [31, 32, 33]). I.e., one can consider the “averaged” partition function

$$Z = \int d\lambda^k \frac{1}{\sqrt{2\pi} \sigma} \exp\left\{-\frac{1}{2\sigma^2} \lambda_k^2\right\} < e^{\lambda^k \int V_k} >.$$

The correlator on the right-hand side represents the partition function of the original theory perturbed by λ_k . It is assumed that σ^2 is of order κ^2 . Performing the integral over λ^k yields

$$Z = < \exp\left\{\frac{\sigma^2}{2} \int V_k \int V_k\right\} > \sim < 1 + \frac{\sigma^2}{2} \int V_k \int V_k + O(\kappa^4) >. \quad (3.6)$$

⁶While this paper was being completed, an interesting preprint [30] appeared in which - in a similar situation - bilocal divergences containing the fictitious cutoff ($\log \hat{a}$) and a cancellation of the corresponding dependence on the fictitious scale are discussed.

If one now introduces a “running width” with some initial value σ_0 ,

$$\sigma^2 \equiv \sigma^2(\tau) \sim \left(\sigma_0^2 - \tau \frac{2}{\alpha} \kappa_0^2 \right) e^{\frac{2Q}{\alpha}\tau}, \quad (3.7)$$

then (3.6) is independent of physical scale transformations; in particular, the τ -dependence of the second term in (3.7) cancels that of (3.5). Thus, the bilocal insertion (3.5) is absorbed by letting the distribution of theories spread under scale transformations in the direction of isolated massless modes.⁷ More precisely, it spreads towards the infrared, corresponding to decreasing τ .

This example illustrates that in the presence of isolated massless modes and handles there is no “classical” renormalization group trajectory $\lambda^k(\tau)$ that describes the same theory at different scales. Instead, averages, or “wave packets” of theories must be considered and the parameters that characterize their shape can also “run”. At higher orders in κ, λ we expect new bi- and multilocal scale dependence, signaling a more complicated “running shape”.

That there is no classical renormalization group trajectory might in fact have been guessed from the analogy between the flow on genus-zero random surfaces and classical string theory that was discussed in the previous section: since handles correspond to string loops, the flow on higher genus surfaces should consequently be described by quantum string theory. The models discussed here have a discrete set of states; therefore one arrives at quantum mechanics rather than field theory. In more general models, the flow should be described by an effective quantum field theory for massless modes.

Having convinced ourselves that this “quantization” of the renormalization group flow is indeed necessary, namely - to $O(\kappa^2)$ - in order to absorb the scale dependence of the bilocal operator insertion (3.5), we can now fix the proportionality constant in front of this insertion to be 1. This ensures that the wave packet spreads at the rate expected from quantum mechanics, $\sigma^2 \sim \hbar t \sim \kappa^2 \phi$. $\sigma^2(\tau)$ in (3.7) then obeys the flow equation

$$\frac{\alpha^2}{4} \ddot{\sigma}^2 - \frac{\alpha}{2} Q \dot{\sigma}^2 = Q \kappa^2.$$

Comparing with (2.16), one can regard $Q\kappa^2$ as the beta function for the (width)².

It should be emphasized that in the presence of gravity the operators that produce bilocal logarithmic divergences are *not* the marginal operators of the matter theory with weight $h_k = 2$, as one might have expected, but those with shifted weight (3.3). In particular, because of this shift the radius R in the $c = 1$ model on a circle coupled to gravity is a superselection parameter: the wave packet does not spread out over R , although the corresponding operator $(\partial x)^2$ is marginal ($h = 2$). This “failure of background independence” has been noted in the matrix models [28].

It is also noteworthy that the scale dependence of (3.5) is linear in τ and does not behave like $\sqrt{\tau}$, as one might have expected since one momentum degree of freedom (the Liouville momentum) is integrated over.

⁷on top of the ordinary $e^{\frac{2Q}{\alpha}\tau}$ dependence of $(\text{sigma})^2$ due to the running κ^2

4. Node III and loop-corrected beta functions

We now turn to node III, which is well-known to modify the string equations of motion - this is the Fischler-Susskind mechanism [16] which has been amply discussed in the literature (see e.g., [17, 34, 35, 36]). Here we only discuss its implications for the flow in the $c = 1$ model on a circle and point out some indirect matrix model evidence for this mechanism. In the next section its relation with the higher-genus string field vertices is pointed out.

4.1. The Fischler-Susskind effect in the $c = 1$ model

Node III can be regarded as node I with the three-punctured sphere replaced by a one-punctured torus (see the drawings in the introduction). If the matter theory has a marginal operator Φ_i , then in analogy with (2.8) the corresponding state $|i\rangle$ propagating through node III yields the logarithmically divergent local operator insertion

$$- (\log \hat{a}) \pi \kappa^2 \rho^i \int \bar{V}_i \quad (4.1)$$

on the surface below, where the operators $V_i \sim \Phi_i$, $\bar{V}_i \sim \Phi_i e^{-Q\phi}$ are normalized to have unit two-point function on the sphere and the genus-one one-point function

$$\rho^i = \frac{1}{v} < \int V_i >_{g=1} \quad (4.2)$$

is the analog of the OPE coefficient c_{jk}^i in the case of node I. v is the integral over possible zero modes of ϕ and the matter fields on the torus.⁸ Since the torus adds the amount $-Q$ to the Liouville background charge the induced operator in (4.1) is the “wrongly dressed” one \bar{V}_i .

In the $c \leq 1$ models, the only nonzero ρ_i occurs for the (normalized) operator $V \equiv \frac{1}{2\pi\alpha'} (\partial x)^2$ in the $c = 1$ model (at higher genus, ρ_i is also nonzero for the operator $R^{(2)}$; see below). If x is compactified on a circle of radius R one finds, setting $\alpha' = 2$ as in section 2:

$$\rho = -\frac{1}{48\pi} \left(1 - \frac{2}{R^2}\right). \quad (4.3)$$

Note that $\rho = 0$ at the self-dual radius $R = \sqrt{2}$. To obtain (4.3), one observes that

$$< \frac{1}{2\pi\alpha'} \int (\partial x)^2 >_{g=1} = -R \frac{\partial}{\partial R} Z_{g=1}. \quad (4.4)$$

This can be seen by redefining $x \rightarrow y = x/R$ in the torus partition function (proportional to the negative free energy) [29]

$$Z_{g=1} = \int \mathcal{D}x \exp\left\{-\frac{1}{4\pi\alpha'} \int (\partial x)^2 + \dots\right\} = \frac{1}{12\sqrt{2}} \left(\frac{R}{\sqrt{\alpha'}} + \frac{\sqrt{\alpha'}}{R}\right) \left|\frac{\log \mu}{\alpha}\right|.$$

⁸The zero mode integrals are overall integrals that must be divided out: the path integral factorizes into integrals over fields $x(\sigma_1)$ on the main surface Σ_1 , fields $x(\sigma_2)$ on the surface Σ_2 that splits off and the zero-mode x_0 : $\mathcal{D}x \rightarrow \mathcal{D}x(\sigma_1) \mathcal{D}x(\sigma_2) dx_0$. Only the integral over $x(\sigma_2)$ is performed in replacing Σ_2 by an operator insertion.

The volumes of the ϕ and x zero modes in this model are given by

$$v = \left| \frac{\log \mu}{\alpha} \right| \times 2\pi R. \quad (4.5)$$

Let us for now ignore node II, which will be included in the next section. (4.1) spoils the background scale invariance of the world-sheet theory. Similarly as in the case of node I, scale invariance must be restored by adding a term of the form $\phi \bar{V}$ to the world-sheet action. In the $c = 1$ case this yields in analogy with (2.10) the kinetic term for x to order κ^2 ,

$$\lambda_0 \frac{(\partial x)^2}{4\pi} - \frac{\pi}{Q} \kappa^2 \rho \frac{(\partial x)^2}{4\pi} \phi e^{-Q\phi} \quad (4.6)$$

with $\lambda_0 = \frac{1}{2}$. The minus sign arises, because $(L_0 + \bar{L}_0 - 2) \phi \bar{V}_i = +2\omega_i \bar{V}_i$, as opposed to (2.9). This sigma model background solves the string equations of motion with cosmological constant [16].⁹ Constant shifts $\phi \rightarrow \phi + \frac{2}{\alpha}\tau$ can now be absorbed in

$$\kappa^2(\tau) = e^{\frac{2Q}{\alpha}\tau} \kappa_0^2, \quad \lambda(\tau) = \lambda_0 + \frac{2}{\alpha Q} \pi \rho \kappa^2(\tau) \tau, \quad (4.7)$$

valid near $\lambda \sim \frac{1}{2}$. Note that the factor $2/(\alpha Q)$ differs from that in (2.12) only by the minus sign. There is no simple first order equation for $\lambda(\tau)$, but it obeys a second order flow equation analogous to (2.16) with $c_{jk}^i \lambda^j \lambda^k$ replaced by $\rho^i \kappa^2$:

$$\frac{\alpha^2}{4} \ddot{\lambda} - \frac{\alpha}{2} Q \dot{\lambda} = \pi \rho \kappa^2. \quad (4.8)$$

This describes a damped motion towards the ultraviolet (remember that $\alpha < 0$), or equivalently an antidamped motion towards the infrared. Instead of keeping the radius R fixed and letting λ run, it is more illuminating to absorb the change in $\lambda(\tau)$ in a redefinition of x . Then $R \rightarrow R\sqrt{\lambda(\tau)/\lambda_0}$ becomes τ -dependent: using (4.7) and (4.3), one finds the flow equation

$$\frac{\alpha^2}{4} \ddot{R} - \frac{\alpha}{2} Q \dot{R} = -\frac{1}{48} \left(R - \frac{2}{R}\right) \kappa^2 + O(\kappa^4).$$

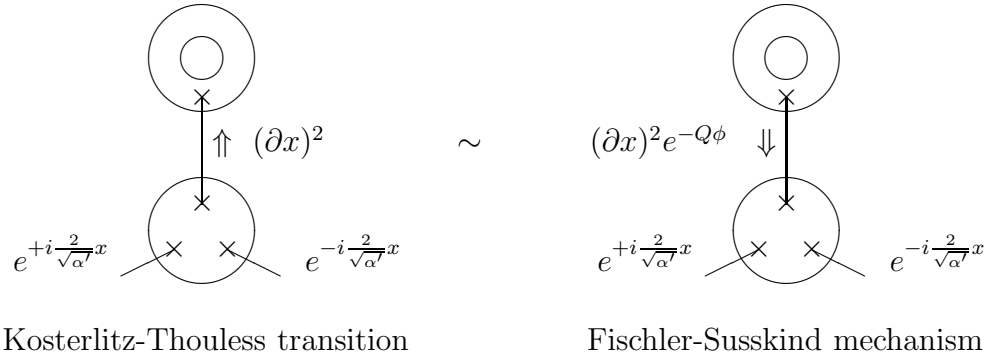
This equation shows that there is a fixed point at the self-dual radius $R = \sqrt{2}$. Linearizing around it, $R = \sqrt{2} + r, r \ll 1$, one finds that this fixed point is unstable in the infrared ($\tau \rightarrow -\infty$): the corresponding beta function (i.e., the right-hand side) is $-\frac{1}{24}\kappa^2 r$, so r is relevant (the fact that $\kappa^2 = \kappa^2(\tau)$ does not change this conclusion). Here it is assumed that vortices are suppressed. Vortices, if allowed, give additional contributions to the beta functions [20]; it is hard to compare them with the $\rho \kappa^2$ term, since κ^2 grows exponentially.

4.2. Comparison with the matrix model

One might worry that - due to subtleties of Liouville theory - perhaps Fischler-Susskind mechanisms are absent in the $c = 1$ model coupled to gravity. Is it possible to confirm their

⁹Note that this background does *not* describe a black hole, despite of its similarity to the black-hole operator $(\partial x)^2 e^{-Q\phi}$. Indeed, the ADM mass is zero [37].

presence in the $c = 1$ matrix model? In the matrix model, the Fischler-Susskind effect should show up as a genus-1 effect, i.e. at first order in the double scaling variable. Therefore one should not expect to see the flow of the radius directly in the genus expansion of the matrix models: instead of observing a state $|(\partial x)^2 e^{-Q\phi} >$ propagating through node III, one should observe the state $|(\partial x)^2 >$ propagating in the opposite direction, corresponding to node I. Now, suppose two marginal external tachyon operators $\exp(\pm i \frac{2}{\sqrt{\alpha'}} x)$ (whose OPE contains the operator $(\partial x)^2$) are inserted into the sphere as drawn below. This is just the situation in which the propagation of the state $|(\partial x)^2 >$ through the node leads to the Kosterlitz-Thouless transition on the torus! It has indeed been observed in the matrix models that - not surprisingly - this transition takes place on surfaces of arbitrary genus [20, 21].



More generally, in the matrix model results Fischler-Susskind effects should be manifest in the form of poles in the higher-genus two-point function that are due to the propagation of the on-shell states through node III. If the torus is replaced by a surface of genus greater than one, contributions to the Fischler-Susskind mechanism come not only from the trace of the graviton $|(\partial x)^2 >$ but also from the zero-momentum dilaton $|R^{(2)} >$ [34, 17, 35]. It is indeed possible to confirm the presence of poles due to these states (or, equivalently, their wrongly dressed counterparts) propagating through the node.¹⁰ To this end, consider the poles in the above genus- g two-point function of the (normalized) tachyon in the vicinity of the Kosterlitz-Thouless momentum $p = \frac{2}{\sqrt{\alpha'}}$,

$$G^{(2)}(p, -p) \sim \frac{1}{\epsilon} \sum_g t^{1-g} G_g + \text{finite} \quad \text{with} \quad p = \frac{2}{\sqrt{\alpha'}} + \epsilon .$$

Here $t = (2\beta\mu)^2$ with $\kappa^2 = \pi^3\beta^2$ has been defined. We now switch to $\alpha' = 1$ to conform with the notation of ref. [29], from which we obtain the residues of the $1/\epsilon$ poles:¹¹

$$G_1 = -\frac{1}{12} \left(R - \frac{1}{R}\right) |\log \mu| \quad (4.9)$$

¹⁰I thank Igor Klebanov for pointing this out to me.

¹¹These poles are present in the matrix model results, though “hidden” in the sense that they are exactly cancelled by poles at discrete tachyon momenta as described in [39]. This cancellation can be understood as a combination of a coupling constant redefinition that is singular at discrete momenta and of the terms (2.6). It does not affect the conclusions of the present discussion.

$$G_2 = \frac{1}{6!} \left(-21R - \frac{10}{R} + \frac{7}{R^3} \right) \quad (4.10)$$

$$G_3 = \frac{1}{7!} \left(-155R - \frac{147}{R} - \frac{49}{R^3} + \frac{31}{R^5} \right). \quad (4.11)$$

One now easily checks that [38]

$$G_g = \left(-R \frac{\partial}{\partial R} + \chi \right) Z_g, \quad (4.12)$$

where

$$\chi = \frac{1}{4\pi} \int d^2\xi \sqrt{g} R^{(2)} = 2 - 2g \quad (4.13)$$

is the Euler characteristic and Z_g is the genus- g partition function ($Z = \sum_g t^{1-g} Z_g$) [29]:

$$Z_1 = \frac{1}{12} \left(R + \frac{1}{R} \right) |\log \mu| \quad (4.14)$$

$$Z_2 = \frac{1}{6!} \left(7R + \frac{10}{R} + \frac{7}{R^3} \right) \quad (4.15)$$

$$Z_3 = \frac{1}{7!} \left(31R + \frac{49}{R} + \frac{49}{R^3} + \frac{31}{R^5} \right). \quad (4.16)$$

In (4.12), $-R \frac{\partial}{\partial R} Z$ corresponds to the genus- g one-point function of the trace of the graviton (see (4.4)), while χZ corresponds to the genus- g one-point function of the zero momentum dilaton (see (4.13)). In this way the residues of the poles in the tachyon two-point function can indeed be identified as the contributions from these two states propagating through the node connecting the genus- g surface with the sphere containing the tachyons. The fact that the R -dependence of the residues precisely works out confirms indirectly that Fischler-Susskind mechanisms work according to the theory in the $c = 1$ model coupled to gravity.

5. Relation with closed string field theory

Let us now assemble pieces of the previous discussion to formally describe the flow on random surfaces with handles in terms of quantum mechanics in theory space, with a potential that is corrected order by order in genus by the elementary higher-genus vertices of closed string field theory.¹²

5.1. Canonical formalism

To order $\kappa^2 \lambda^0$, it has already been seen in section 3 that ‘quantization’ of the flow is necessary in the case of massless λ^i in order to absorb the scale dependence induced by node II. At higher orders in λ it will be useful to quantize *all* coupling constants. By this, the following is meant. Consider first the operator insertions (3.1). As is well-known (see, e.g., [32]),

¹²For other discussions of relations between the flow and string field theory see [31, 33] and [40]. Different conclusions about renormalization group flows, string theory and other issues can be found in [41].

such bilocal expressions can be reproduced by turning the coupling constants into quantum operators. More precisely, consider first the linearized flow equations. They can be regarded as the linearized string equations of motion for $\lambda^i(\phi)$ as discussed in subsection 2.2 (here, a dot means derivative with respect to $\phi \sim -\frac{2}{\alpha}\tau$):

$$\ddot{\lambda}^k + Q\dot{\lambda}^k = \beta^k = (h_k - 2)\lambda^k. \quad (5.1)$$

These equations can be transformed into the familiar (euclidean) harmonic oscillator equations by redefining

$$\chi^k \equiv e^{\frac{Q}{2}\phi} \lambda^k \quad \rightarrow \quad \ddot{\chi}^k = (h_k - 2 + \frac{Q^2}{4})\chi^k = \omega_k^2 \chi^k.$$

The coupling constants λ^i (or χ^i) can now be replaced by operators $\hat{\lambda}^i$ with free mode expansion

$$\hat{\lambda}_k^{free}(\phi) \sim \hat{a}_k^\dagger e^{\alpha_k^+ \phi} + \hat{a}_k e^{\alpha_k^- \phi}.$$

$\hat{a}_k^\dagger, \hat{a}_k$ are creation and annihilation operators with commutation relations

$$[\hat{a}_i, \hat{a}_j^\dagger] \sim \frac{\kappa^2}{2\omega_i} \delta_{ij}, \quad [\hat{a}_i, \hat{a}_j] = [\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0.$$

\hat{a}^\dagger and \hat{a} create and annihilate strings. If correlation functions are sandwiched between in- and out-vacua $|0\rangle$ and $\langle 0|$, defined by $\hat{a}_k|0\rangle = 0$ and $\langle 0|\hat{a}_k^\dagger = 0$, then the insertions (3.1) are reproduced by contractions of the \hat{a}_k 's and \hat{a}_k^\dagger 's in

$$\langle \langle 0| \exp\left\{ \int \hat{\lambda}_i^{free}(\phi) \Phi_i \right\} V_1 \dots V_n |0\rangle \rangle \quad (5.2)$$

$$\sim \langle \langle 0| \exp\left\{ \hat{a}_i^\dagger \int V_i + \hat{a}_i \int \bar{V}_i \right\} V_1 \dots V_n |0\rangle \rangle. \quad (5.3)$$

The meaning of (5.3) can be made clear in the Schrödinger picture: “Integrating out handles” amounts to introducing random fluctuations of the coupling constants around the fixed point, with a Gaussian distribution of width $\sigma \sim (\kappa/\sqrt{\omega_i})$. Unlike in the massless case $\omega_i \rightarrow 0$, for $\omega_i > 0$ this distribution does not spread under scale transformations, apart from the overall spread that can be absorbed in the running topological coupling constant κ^2 .

Note that this “quantization” procedure of the flow is also in accord with the matrix model results of [42]. There it was shown that indeed the torus free energy of a (p, q) minimal model can - after zeta function regularization and dividing out by the Liouville volume - be written as the sum over the (real time) harmonic oscillator ground state energies of all the modes λ^k with the same ω_i that have been defined above:

$$F \sim \sum_i \frac{\omega_i}{2} = -\frac{(p-1)(q-1)}{24(p+q-1)}.$$

In view of the interpretation of time ϕ as the scale factor, this could be regarded as a “zero-point central charge” of conformal field theories in the presence of handles.

Now, in the presence of interactions c_{jk}^i , the quadratic beta function coefficients in (2.16) must be included in the equations of motion (5.1). Solving them yields the $O(\kappa^2\lambda)$ generalization of (5.3), dropping the operator insertions (compare with (2.6)):

$$\langle \langle 0 | \exp \{ \hat{a}_i^\dagger \int V_i + \hat{a}_i \int \bar{V}_i + \pi \frac{c_{ik}^j}{2\omega_j} \hat{a}^{i\dagger} \hat{a}^{k\dagger} \int \phi V_j - \pi \frac{c_{i\bar{j}}^{\bar{k}}}{2\omega_k} \hat{a}^j \hat{a}^{i\dagger} \int \phi \bar{V}_k + \dots \} | 0 \rangle \rangle. \quad (5.4)$$

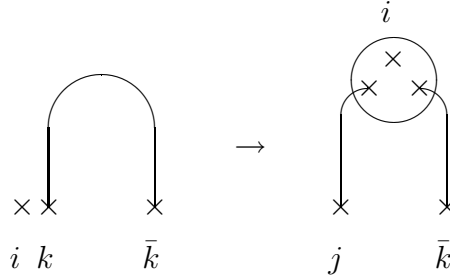
Here it has been taken into account that there are nontrivial OPE coefficients

$$V_i(z) \bar{V}_j(0) \sim \frac{1}{|z|^2} c_{i\bar{j}}^{\bar{k}} \bar{V}_k + \dots \quad \text{with} \quad c_{i\bar{j}}^{\bar{k}} = c_{ik}^j.$$

It is interesting how (5.4) manages to be independent of the background scale $\sqrt{\hat{g}}$. Consider, e.g., the derivative of (5.4) with respect to $\hat{a}^{i\dagger}$,

$$\langle \int V_i + \pi \frac{\kappa^2}{2\omega_k} \frac{c_{ik}^j}{2\omega_j} \int \phi V_j \int \bar{V}_k - (j \leftrightarrow k) + \text{other contractions} \rangle. \quad (5.5)$$

The second and third terms come from the contraction of linear and quadratic terms in the exponential. Now, V_i in (5.5) may collide with one of the operators of the bilocal insertions (3.1) (which are produced by contractions of the linear terms in (5.4)):



This situation leads to the standard pole in string amplitudes that is due to the “pair production” of “particle” j and “anti-particle” \bar{k} . On the world-sheet, this pole shows up as a logarithmically divergent bilocal operator insertion that depends on the fictitious scale through the cutoff \hat{a} :

$$\pi (\log \hat{a}) \frac{\kappa^2}{2\omega_k} \frac{c_{ik}^j}{2\omega_j} \int V_j \int \bar{V}_k + (j \leftrightarrow k). \quad (5.6)$$

It can be found by either integrating over the moduli or by using the OPE of V_k with V_i . Using (2.9) (and its counterpart for \bar{V}), one sees that the scale dependence of (5.6) is now precisely cancelled by that of the second and third terms in (5.5).

5.2. Closed string field vertices

We can now show how the higher-genus string field vertices formally enter the flow equations. The point is that part of node III corresponds to the situation where the two ends of node

II coincide with each other, forming a three-vertex with two legs connected by a propagator (see figure below). The corresponding scale dependence is *already* cancelled if the λ^i are turned into operators in order to account for node II. Namely, in (5.4) the contraction of the last term with itself corresponds to inserting a propagator on the 3-vertex. This yields already at order $\kappa^2 \lambda^0$ an insertion

$$-\sum_{i,j,k} \frac{\pi}{2\omega_k} c_{ij}^{\bar{k}} < 0 | \hat{a}_j \hat{a}_i^\dagger | 0 > < \int \phi \bar{V}_k > \sim -\sum_{i,k} \frac{\pi \kappa^2}{4\omega_i \omega_k} c_{ik}^i < \int \phi \bar{V}_k > .$$

Since $(L_0 - \bar{L}_0 - 2)\phi \bar{V}_k = -2\omega_k V_k$, this cancels part of the scale dependence (4.1) that is due to node III. Only the scale dependence from the remainder of node III,

$$(\nu_{1,1})_i \sim \rho_i + \sum_l \frac{1}{2\omega_l} c_{il}^l,$$

must then be cancelled by adding an $O(\kappa^2)$ term to the beta functions as in (4.8). $\nu_{1,1}$ is the elementary genus-1 1-vertex of closed string field theory [26] - the part of the moduli space of a one-punctured torus that cannot be obtained by connecting a genus-zero vertex with a propagator.

$$\rho_i = -\frac{1}{2\omega_l} c_{il}^l + (\nu_{1,1})_i$$

The flow equations (2.16) finally become, after rescaling the vertices (for almost marginal operators with dimensions $h_i \sim 2$):

$$\frac{\alpha^2}{4} \ddot{\lambda}^i - \frac{\alpha}{2} Q \dot{\lambda}^i = (h_i - 2) \hat{\lambda}^i + \pi c_{jk}^i \hat{\lambda}^j \hat{\lambda}^k + \pi d_{jkl}^i \hat{\lambda}^j \hat{\lambda}^k \hat{\lambda}^l + \kappa^2 (\nu_{1,1})^i + \dots \quad (5.7)$$

In fact, c_{jk}^i and d_{jkl}^i can be thought of as the genus-0 string field vertices $\nu_{0,3}$ and $\nu_{0,4}$. Those are the genus-0 3-point function and the part of the genus-0 4-point function that cannot be built by connecting two 3-point functions with a propagator (the other part is already taken care of by solving (5.7) for λ^k to order λ^2 and then plugging the result back into the quadratic term). In this sense the flow is described, at least to this order, by canonically quantized closed string field theory.

More generally, turning the coupling constants λ^i into operators should absorb all scale dependence that arises from handles that are added to tree diagrams (nodes II). The remaining scale dependence must be cancelled by hand with the help of the elementary genus- g N -vertices $\nu_{g,N}$. At higher orders in λ, κ those may also modify the time derivative operator in (5.7), as noted at the end of section 2. Of course, a complete treatment of the BRST cohomology should involve the full BV formalism as in [26].

6. Speculation and summary

6.1. Speculation for $\hat{c} \geq 9$

In the previous sections, the vicinity of fixed points with central charge $c \leq 1$ has been discussed. Let us now suggest how our discussion should be modified for models with $c \geq 25$ – or rather, for supersymmetric theories with $\hat{c} \geq 9$ coupled to supergravity, with the tachyon projected out.¹³ The fixed points with $\hat{c} \geq 9$ can, e.g., be sigma models with Euclidean signature; the conformal factor then becomes a timelike target space coordinate.

1. Concerning node II, since these theories contain full-fledged target space fields, the flow should be described by quantum field theory rather than quantum mechanics. Concerning node III, the genus-one tadpole effects should cancel due to supersymmetry.
2. In analogy with [15] for the case $c \geq 25$, we expect that for $\hat{c} \geq 9$ the flow to the infrared is damped rather than anti-damped. This is a result of the fact that time is minkowskian for $\hat{c} \geq 9$. As a consequence, *both* Liouville-dressings converge at an infrared fixed point and both correspond to equally good world-sheet operators, at least in the infrared region. Presumably this means that the general renormalization group trajectory is a general string solution, which contains both dressings. In the $c \leq 1$ models by contrast, the “wrong” Liouville dressing which corresponds to the divergent classical solution is suppressed in the path integral (as is usual for euclidean quantum mechanics). In theories on a fixed lattice, of course, the coupling constants at one scale must be uniquely determined by the coupling constants at a different scale. But in our case the scale is a dynamical variable, and it is no longer clear that a *unique* renormalization group trajectory must pass through a given point in theory space.
3. Before settling down at an infrared fixed point, renormalization group trajectories may oscillate around it. If the central charge of the infrared fixed point is $\hat{c} = 9$, we expect in analogy with an example in [15] that the oscillations are no longer exponentially damped but decay much slower in time, $\sim \frac{1}{t}$. They correspond to wave-like excitations of the target space fields.
4. Furthermore, we expect that there are two disconnected sectors of string solutions or renormalization group trajectories - those corresponding to euclidean and minkowskian target space signature. Static trajectories corresponding to fixed points with $\hat{c} > 9$ are in the latter sector, while those corresponding to fixed points with $\hat{c} < 9$ are in the former sector.¹⁴ (fixed points with $\hat{c} = 9$ can belong to both sectors, depending on how they are approached). If a theory starts from an ultraviolet fixed point with $\hat{c} \geq 9$, it cannot converge towards an infrared fixed point with $\hat{c} < 9$, as in [15]. One may speculate that the quantum analog of this statement is that string vacua with $\hat{c} = 9$ cannot decay.

¹³For a discussion of the gravitational dressing of one-loop beta functions in $N = 1$ supersymmetric theories with $\hat{c} \leq 1$ (and of the absence of such a dressing in $N = 2$ theories) see [9].

¹⁴Such static trajectories correspond to the cosmological solutions of [43].

5. What are the infrared stable fixed points with $\hat{c} \geq 9$? Apparently they must obey two conditions. First there must be no relevant operators in the matter theory, or equivalently no tachyons in the string theory; otherwise the fixed point is not infrared stable. Second, it seems that the matter theory must be modular invariant because the moduli are integrated over - i.e., any theory coupled to gravity should at least be equivalent to a modular invariant theory coupled to gravity. Combining both conditions, one concludes that the infrared fixed points correspond to consistent string theories!

In view of this, it is of course very tempting to regard closed string field theory as the theory of the flow in the most general unitary theory that lives on a (super) random surface. One might speculate that this is a continuum limit of some unknown statistical mechanical system - perhaps a $\hat{c} \geq 9$ analog of the ensemble of large Feynman graphs of the matrix models. Like any such system, it would flow to an infrared fixed point - a string vacuum. Since renormalization group time is identified with real time in target space, this flow would have the interpretation of a cosmological evolution.

In the vicinity of fixed points, it would make sense to describe the system as a target space field theory. If the infrared fixed point had $\hat{c} = 9$, perhaps after tunneling down from $\hat{c} \geq 9$, there would be oscillations of the flow around it. Their spectrum could be compared with the observed spectrum of elementary particles. It must be pointed out, though, that if one took this seriously, one would have to assume that the underlying statistical mechanical system is huge: since time corresponds to the world-sheet scale, we would presently be observing this system at scales of the order of at least

$$10^{10^{61}}$$

times its cutoff-scale, where 10^{61} is the age of the universe in Planck units, and in fact this number would be growing fast as “scale” goes by.

6.2. Summary

Let us summarize the effects of gravity on the renormalization group flow that have been discussed here. It has been seen that, due to fluctuations of the conformal factor, at least for $c < 1$ and up to cubic order in coupling constants, the time derivative in the standard flow equation (1.2) is simply replaced by the second-order derivative operator

$$\dot{\lambda}^i \rightarrow \frac{\alpha^2}{4} \ddot{\lambda}^i - \frac{\alpha}{2} Q \dot{\lambda}^i .$$

To this order, this turns the flow equations into the string equations of motion. The following variations of this general theme of second-order flow equations have been discussed (a dot means derivative with respect to $-\frac{\alpha}{2}\phi$):

First, for (almost) marginal operators the flow equations become

$$\frac{\alpha^2}{4} \ddot{\lambda}^i - \frac{\alpha}{2} Q \dot{\lambda}^i = c_{jk}^i \lambda^j \lambda^k + d_{jkl}^i \lambda^i \lambda^j \lambda^k + \dots$$

Picking the solution that also obeys a standard first-order equation yields the gravitational dressing (2.18) of the beta function coefficients c_{jk}^i and d_{jkl}^i for almost marginal operators. The result for the dressing of c_{jk}^i agrees with the light-cone gauge result. It would be interesting to also compute the dressing of d_{jkl}^i in light-cone gauge.

Second, in the presence of handles the topological coupling constant κ^2 runs as in (3.2). To lowest order in κ^2 it obeys the flow equation

$$\frac{\alpha^2}{4}\ddot{\kappa}^2 - \frac{\alpha}{2}Q\dot{\kappa}^2 = 0.$$

Furthermore, a curious phenomenon takes place if the matter theory has isolated “massless” states with dimension $h_i = 2 - \frac{Q^2}{4}$. Due to pinched handles, the ‘classical’ renormalization group trajectory $\lambda^i(\tau)$ is then replaced by a “wave packet” of theories that spreads under scale transformations in the directions corresponding to these λ^i . Its width square σ^2 obeys the flow equation

$$\frac{\alpha^2}{4}\ddot{\sigma}^2 - \frac{\alpha}{2}Q\dot{\sigma}^2 = Q\kappa^2$$

with ‘beta function’ $Q\kappa^2$. More generally, in the presence of handles distributions of theories must be considered instead of points in theory space. The moments that describe their shape also become running coupling constants with flow equations analogous to those for the λ ’s. Effectively, the flow is then described by a quantum theory for massless modes. One wonders whether there are related phenomena in solid state physics or statistical mechanics.

Third, beta functions are modified by Fischler-Susskind effects. E.g., in the $c = 1$ model on a circle the radius becomes a running coupling constant, obeying the equation ($\alpha' = 2$):

$$\frac{\alpha^2}{4}\ddot{R} - \frac{\alpha}{2}Q\dot{R} = -\frac{1}{48}\kappa^2(R - \frac{2}{R}).$$

The self-dual radius corresponds to an ultraviolet stable fixed point. The poles associated with the Fischler-Susskind mechanism that are due to the trace of the graviton and the zero-momentum dilaton propagating through node III precisely agree with the matrix model results at least up to genus 3. After “quantizing” the flow to account for node II, i.e., after replacing the coupling constants by operators, the loop corrections to the beta functions reduce to the elementary vertices $\nu_{g,N}$ of closed string field theory:

$$\frac{\alpha^2}{4}\ddot{\hat{\lambda}}^i - \frac{\alpha}{2}Q\dot{\hat{\lambda}}^i \sim \beta^i + \kappa^2(\nu_{1,1})^i + \kappa^2(\nu_{1,2})_j^i \hat{\lambda}^j + \dots$$

(only the case $\nu_{1,1}$ has been discussed explicitly. At higher orders, the left-hand side might also be corrected). This is in accord with the assertion that the renormalization group flow in theories that live on random surfaces with handles is described by closed string field theory. One might have thought that the theory of the flow ins only “half of string theory”, since the solutions with negative Liouville dressing are forbidden; but if the argument in point 2 of the previous subsection is correct, both dressings should be allowed in the regime $\hat{c} \geq 9$, where the flow to the infrared is damped rather than anti-damped.

A very interesting question is whether the flow in the presence of handles can also tunnel between different infrared fixed points (as has been speculated in the last subsection). One

could, e.g., imagine that such tunneling induces a transition from the $c = 1$ phase to the $c = 0$ phase of the $O(2)$ model on a random surface. Perhaps this can be checked in the corresponding matrix model [20]. It would be a nonperturbative effect of order $\exp(-1/\kappa^2)$. Another challenge of course is to understand the nonperturbative features of order $\exp(-1/\kappa)$ of the flow between minimal models that have been observed in the matrix models [44].

Finally, let us note that one can write down fixed point conditions for the flow on random surfaces of arbitrary genus by cancelling the scale-dependence from the various components of the boundary of moduli space. They are (somewhat) reminiscent of the Virasoro constraints [45], the master equation for $2d$ string theory [46], and the holomorphic anomaly equation of [47]. It is not clear to the author whether there is any connection.

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